MEASUREMENT WITHOUT ARCHIMEDEAN AXIOMS*

LOUIS NARENS

University of California, Irvine

Axiomatizations of measurement systems usually require an axiom—called an *Archimedean axiom*—that allows quantities to be compared. This type of axiom has a different form from the other measurement axioms, and cannot—except in the most trivial cases—be empirically verified. In this paper, representation theorems for extensive measurement structures without Archimedean axioms are given. Such structures are represented in measurement spaces that are generalizations of the real number system. Furthermore, a precise description of "Archimedean axioms" is given and it is shown that in all interesting cases "Archimedean axioms" are independent of other measurement axioms.

1. Preliminaries. *Notation.* Throughout this paper the following convention will be observed. Re will stand for the real numbers; Re^+ for the positive real numbers; I for the set of integers; I^+ for the set of positive integers; and $\langle x_1, \ldots, x_n \rangle$ and $(x_1, \ldots, x_n)$ for ordered n-tuples; *iff* will stand for the phrase 'if and only if'.

Definition 1.1. Let J be a nonempty set and π a function from J into the nonnegative integers. A *relational system of type* π is an ordered pair $\langle A, \mathscr{F} \rangle$ where A is a nonempty set and $\mathscr{F} = \{R_j \mid j \in J\}$ is a family of relations on A such that for each $j \in J$, R_j is a $\pi(j)$-ary relation.

Comments on *Definition 1.1*:

(1) By definition, a 0-ary relation on A is a member of A.

(2) There is no bound on the cardinality of J. In fact, we will often use index sets J of cardinality greater than the continuum.

(3) What is here called 'relational systems' are often elsewhere called 'models (for first order languages)'.

(4) Relations on A that have a special role are often listed separately from other relations. Thus if we are concerned with an ordering relation $\precsim$ on A, $\precsim \in \mathscr{F}$, we may write $\langle A, \mathscr{F} \rangle$ as $\langle A, \precsim, \mathscr{F} \rangle$, etc.

(5) Operations on A can be represented as relations on A. For example, the two place operation $+$ on A can be thought of as the three place relation R on A where $R(a, b, c)$ holds if and only if $a + b = c$.

Definition 1.2. Let A be a nonempty set and $\mathscr{F} = \{R_j \mid j \in J\}$ the set of all relations on A. Then $\langle A, \mathscr{F} \rangle$ is called *the full relational system of A of type* π, where π is the function from J into the nonnegative integers defined by: $\pi(j) = n$ if and only if R_j is a n-ary relation.

Definition 1.3. Let J be a nonempty set and $\mathscr{F} = \{R_j \mid j \in J\}$ and π a function from J into the nonnegative integers. By definition, *the language* $L_\pi(\mathscr{F})$ is the first order language that has $\mathscr{F} = \{\mathbf{R}_j \mid j \in J\}$ as its set of predicate symbols where for each $j \in J$, $\mathbf{R}_j$ is a $\pi(j)$-ary predicate symbol. (In $L_\pi(\mathscr{F})$, $\wedge$ is the conjunction symbol, $\vee$ the disjunction symbol, $\rightarrow$ the implication symbol, $\leftrightarrow$ the "if and only if"

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symbol, and $\neg$ the negation symbol.) If $\langle A, \mathscr{G}\rangle$ is a relational system of type π, then by the usual definition of *truth* for first order languages (see [1], Chapter 3) each sentence of $L_\pi(\mathscr{F})$ is assigned by $\langle A, \mathscr{G}\rangle$ the truth value *true* or the truth value *false*.

Comments on *Definition 1.3*:

(1) 0-place predicate symbols are often called *individual constants* or *individual constant symbols*.

(2) Frequently, 'π' will be omitted from the expression '$L_\pi(\mathscr{F})$'. Thus, the expression '$L(\mathscr{F})$' may be thought of as an abbreviation for '$L_\pi(\mathscr{F})$'.

Definition 1.4. Let Γ be a set of sentences of $L_\pi(\mathscr{F})$ and $\langle A, \mathscr{G}\rangle$ be a relational system of type π. Then $\langle A, \mathscr{G}\rangle$ is said to be *a relational system for* Γ if and only if each sentence of Γ is true in $\langle A, \mathscr{G}\rangle$.

Definition 1.5. Let $\langle A, \mathscr{F}\rangle$ and $\langle B, \mathscr{G}\rangle$ be relational systems of type π. Then $\langle A, \mathscr{F}\rangle$ and $\langle B, \mathscr{G}\rangle$ are said to be *elementarily equivalent* if and only if each sentence of $L_\pi(\mathscr{F})$ that is true in $\langle A, \mathscr{F}\rangle$ is also true in $\langle B, \mathscr{G}\rangle$.

Definition 1.6. Let Σ be a set of sentences of $L(\mathscr{F})$. Σ is said to be *simultaneously satisfiable* if and only if there is a relational system $\langle A, \mathscr{F}\rangle$ of type π such that each sentence of Σ is true in $\langle A, \mathscr{F}\rangle$. Σ is said to be *finitely satisfiable* if and only if each finite subset of Σ is simultaneously satisfiable. If $\langle A, \mathscr{F}\rangle$ is a relational system such that each sentence of Σ is true in $\langle A, \mathscr{F}\rangle$ then $\langle A, \mathscr{F}\rangle$ is said to *simultaneously satisfy* Σ and Σ is said to be *simultaneously satisfiable in* $\langle A, \mathscr{F}\rangle$.

The proof of the following fundamental theorem can be found in [1].

> *Theorem 1.1. The Compactness Theorem of Logic.* If Σ is a set of sentences of $L(\mathscr{F})$ that is finitely satisfiable then Σ is simultaneously satisfiable.

Definition 1.7. Let $\langle A, \precsim, \mathscr{F}\rangle$ be a relational system where $\precsim$ is a binary relation. $\precsim$ is said to be a *weak ordering on A* if and only if the following three sentences of $L(\mathscr{F})$ are true in $\langle A, \precsim, \mathscr{F}\rangle$:

(1) $\forall x(x \precsim x)$,
(2) $\forall x\forall y\forall z((x \precsim y \wedge y \precsim z) \rightarrow x \precsim z)$,
(3) $\forall x\forall y(x \precsim y \vee y \precsim x)$.

Definition 1.8. If $\precsim$ is a weak ordering on A and a, b are elements of A, then, by definition, $a \sim b$ iff $a \precsim b$ and $b \precsim a$. It is easy to show that $\sim$ is an equivalence relation on A. $\precsim$ is said to be a *total ordering* on A iff each equivalence class determined by $\sim$ has exactly one member. By definition, $a \prec b$ iff $a \precsim b$ and not $a \sim b$. Also by definition, $a \succsim b$ iff $b \precsim a$.

Definition 1.9. Let $\langle A, \precsim, \circ\rangle$ be a relational system where $\precsim$ is a binary relation and $\circ$ a binary operation. Then $\langle A, \precsim, \circ\rangle$ is said to be an *ordered abelian group* if and only if it satisfies the following axioms:

(1) $\precsim$ is a weak order on A;
(2) for each x and y in A, $x \circ y \sim y \circ x$;
(3) for each x, y, and z in A, $x \circ (y \circ z) \sim (x \circ y) \circ z$;
(4) for each $x, y \in A$ there is a $z \in A$ such that $x \circ z \sim y$;
(5) if x, y, z, w are in A and $x \precsim y$ and $z \precsim w$ then $x \circ z \precsim y \circ w$.

Naturally, each of the axioms for an ordered abelian group can easily be formulated as a statement of $L(\mathscr{F})$ where $\mathscr{F} = \{\precsim, \circ\}$.

Definition 1.10. Let $\langle A, \precsim, \circ \rangle$ be an ordered abelian group. It is easy to show that there is an unique (up to equivalence) element e in A (called *the identity element*) such that $a \circ e \sim a$ for all a in A. Let, by definition, $A^+ = \{a \in A \mid e \prec a\}$. By definition, $|x|$ is a function from A into A such that (1) if $e \precsim a$ then $|a| = a$ and (2) if $e \succ a$ then $|a| = b$ where b is some element of A such that $a \circ b \sim e$.

2. Archimedean Axioms. *Definition 2.1.* Let $\langle A, \precsim, \circ \rangle$ be an ordered abelian group. Let $a \in A$. Inductively define na for $n \in I^+$ as follows:

(i) $1a = a$,
(ii) $(n + 1)a = (na) \circ a$.

Let $\mathscr{F} = \{\precsim, \circ\}$. In *Definition 2.1* note that 'na' is not immediately formalizable as an expression of $L(\mathscr{F})$ since 'n' is not in the language of $L(\mathscr{F})$. But '$3a$' means '$(a \circ a) \circ a$' and this latter expression is easily formalizable in $L(\mathscr{F})$. In general, 'na' is formalizable in $L(\mathscr{F})$ for each $n \in I^+$ by an expression that becomes increasingly longer for larger n.

Let $\langle A, \mathscr{F} \rangle$ be a relational system of type π. In general, not all properties of $\langle A, \mathscr{F} \rangle$ can be formulated in terms of $L(\mathscr{F})$. For example, if A is an infinite set, it can be shown that "the cardinality of A" cannot be formulated in $L(\mathscr{F})$. In particular, "Archimedean" axioms cannot be formulated in $L(\mathscr{F})$.

Definition 2.2. Let $\langle A, \precsim, \circ \rangle$ be an ordered abelian group. Then $\langle A, \precsim, \circ \rangle$ is said to be *Archimedean* if and only if for each a, b in A^+, if $a \prec b$ then for some $n \in I^+$, $b \precsim na$.

It is interesting to look at the parallel definition of 'Archimedean' using as much of the language $L(\mathscr{F})$ as possible. Let $\langle A, \precsim, \circ, \mathscr{F} \rangle$ be the full relational system of A of type π. Assume that $\langle A, \precsim, \circ \rangle$ is an ordered abelian group. For each $a \in A$ and each $i \in I^+$ let the formula $i\mathbf{a}$ of $L(\mathscr{F})$ be defined inductively as

$$1\mathbf{a} = \mathbf{a},$$
$$(i + 1)\mathbf{a} = ((i\mathbf{a}) \bullet \mathbf{a}).$$

For each a, b in A and each $i \in I^+$, let $\Psi_i(a, b)$ be the following sentence of $L(\mathscr{F})$:

$$\Psi_i(a, b)\colon \mathbf{b} \precsim i\mathbf{a}$$

Then $\langle A, \circ, \precsim \rangle$ is Archimedean if and only if for each $a, b \in A^+$ there is an $i \in I^+$ such that $\Psi_i(a, b)$ is true in $\langle A, \mathscr{F} \rangle$. (Note that 'for each a, b in A^+ . . .' can be formulated in $L(\mathscr{F})$ by '$\forall x \forall y \forall z(z \precsim x \wedge z \precsim y \wedge \forall x_1(z \bullet x_1 \sim x_1) \rightarrow \ldots)$'. However, there is no way of formulating 'there is an $i \in I^+$ such that $\Psi_i \ldots$' in $L(\mathscr{F})$.

In practice, Archimedean axioms take many different forms. We will now give a generalized definition of 'Archimedean axiom' that captures the essential quality of 'Archimedeanness'.

Definition 2.3. Let $\langle A, \mathscr{G} \rangle$ be a relational system and $\langle A, \mathscr{F} \rangle$ a full relational system of A. (Thus $\mathscr{G} \subseteq \mathscr{F}$.) Let $\mathscr{A} = \{\Psi_i(x_1, x_2, \ldots, x_n) \mid i \in I^+\}$ be a set of formulas of $L(\mathscr{F})$. $\mathscr{A}$ is said to be an *Archimedean schemata for* $\langle A, \mathscr{G} \rangle$ if and only if for each $a_1, a_2, \ldots, a_n$ in A there is an $i \in I^+$ such that $\Psi_i(\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n)$ is true in $\langle A, \mathscr{F}\rangle$.

Definition 2.4. Let Γ be a set of sentences of $L(\mathscr{G})$. Let $\mathscr{A} = \{\Psi_i(x_1, \ldots, x_n) \mid i \in I^+\}$ be a set of formulas of $L(\mathscr{G})$. Then $\mathscr{A}$ is said to be a *good Archimedean axiom for* Γ if and only if there is a relational system $\langle A, \mathscr{G}'\rangle$ such that:

(1) $\langle A, \mathscr{G}'\rangle$ is a relational system for Γ, and
(2) $\mathscr{A}$ is an Archimedean schemata for $\langle A, \mathscr{G}'\rangle$.

The following theorem shows that in all interesting situations Archimedean axioms are not derivable from axioms expressible in $L(\mathscr{G})$.

Theorem 2.1. Let Γ be a set of sentences for $L(\mathscr{G})$ and $\mathscr{A} = \{\Psi_i(x_1, \ldots, x_n) \mid i \in I^+\}$ be a good Archimedean axiom for Γ. Suppose that for each $j \in I^+$ there is a relational system $\langle A_j, \mathscr{G}_j\rangle$ of Γ such that there are elements $a_1{}^j, a_2{}^j, \ldots, a_n{}^j$ of A_j such that if $\langle A_j, \mathscr{F}_j\rangle$ is the full relational system of A_j then for all $i \le j$, $\neg\, \Psi_i(\mathbf{a}_1{}^j, \ldots, \mathbf{a}_n{}^j)$ is true in $\langle A_j, \mathscr{F}_j\rangle$. Then there is a relational system $\langle A, \mathscr{G}'\rangle$ of Γ for which the Archimedean axiom $\mathscr{A}$ fails, i.e. there are $a_1, \ldots, a_n$ of A such that for each $i \in I^+$ $\neg\, \Psi_i(\mathbf{a}_1, \ldots, \mathbf{a}_n)$ is true in the full relational system of A.

Proof. Let $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n$ be new constant symbols that are not in the language $L(\mathscr{G})$. Let $\Gamma' = \{\neg\, \Psi_i(\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n) \mid i \in I^+\}$. We will show that $\Sigma = \Gamma \cup \Gamma'$ is finitely satisfiable. Let $\theta_1, \theta_2, \ldots, \theta_n$ be finitely many statements of Σ. Without loss of generality suppose that $\theta_1, \theta_2, \ldots, \theta_k$ are members of Γ and $\theta_{k+1}, \ldots, \theta_n$ are members of Γ'. Since for $k + 1 \le i \le n$, θ_i is $\neg\, \Psi_{m_i}(\mathbf{b}_1, \ldots, \mathbf{b}_n)$ for some m_i, let m be the maximum member of the set $\{m_i \mid k + 1 \le i \le n\}$. By hypothesis, let $\langle A_m, \mathscr{G}_m\rangle$ and $a_1{}^m, \ldots, a_n{}^m$ be such that

(1) $\langle A_m, \mathscr{G}_m\rangle$ is a relational system for Γ, and
(2) $a_1{}^m, \ldots, a_n{}^m$ are in A_m and that for all $i \le m$

$\neg\, \Psi_i(\mathbf{a}_1{}^m, \ldots, \mathbf{a}_n{}^m)$ is true in the full relational system of A_m. Interpret $\mathbf{b}_1$ as $a_1{}^m$, $\mathbf{b}_2$ as $a_2{}^m, \ldots, \mathbf{b}_n$ as $a_n{}^m$. Then $\theta_1, \theta_2, \ldots, \theta_n$ are true in the full relational system of A_m. Thus Σ is finitely satisfiable. By the compactness theorem (*Theorem 1.1*), let $\langle A, \mathscr{G}'\rangle$ be a relational system in which each sentence of Σ is true. Let $a_1, \ldots, a_n$ be the interpretations of $\mathbf{b}_1, \ldots, \mathbf{b}_n$ in $\langle A, \mathscr{G}'\rangle$. Then since for each $i \in I^+$ $\neg\, \Psi_i(\mathbf{b}_1, \ldots, \mathbf{b}_n)$ is true in $\langle A, \mathscr{G}'\rangle$, $\neg\, \Psi_i(\mathbf{a}_1, \ldots, \mathbf{a}_n)$ is also true in $\langle A, \mathscr{G}'\rangle$. Thus $\mathscr{A}$ fails in $\langle A, \mathscr{G}'\rangle$.

3. Nonstandard Models of the Reals. *Definition 3.1.* Let $\langle Re, \mathscr{F}\rangle$ be the full relational system of Re. $\langle *Re, *\mathscr{F}\rangle$ is said to be a *nonstandard model of the reals* if and only if the following three conditions hold:

(1) $\langle Re, \mathscr{F}\rangle$ and $\langle *Re, *\mathscr{F}\rangle$ are elementarily equivalent in the language $L(\mathscr{F})$;
(2) if $a \in Re$ then $\mathbf{a}$ is interpreted in $\langle *Re, *\mathscr{F}\rangle$ as a; and
(3) Re is a proper subset of $*Re$.

Theorem 3.1. There is a nonstandard model of the reals.

Proof. Let $\langle Re, \mathscr{F}\rangle$ be the full relational system of Re. Let Γ be the set of sentences of $L(\mathscr{F})$ that are true in $\langle Re, \mathscr{F}\rangle$. Let $\mathbf{b}$ be a new constant symbol that is not in $L(\mathscr{F})$. Let Γ' be the following set of sentences: $\Gamma' = \{\mathbf{b} \neq \mathbf{a} \mid a \in Re\}$. Let $\Sigma = \Gamma \cup \Gamma'$. We will show that Σ is finitely satisfiable. Let $\theta_1, \theta_2, \ldots, \theta_n$ be members of Σ. Without loss of generality assume that $\theta_1, \ldots, \theta_k$ are in Γ and $\theta_{k+1}, \ldots, \theta_n$ are in Γ'. Then for each $i, k+1 \leq i \leq n$, θ_i is a sentence of the form $\mathbf{a}_i \neq \mathbf{b}$ where $a_i \in Re$. Since Re is an infinite set, let $b' \in Re$ be such that $b' \neq a_i$ for each $i, k+1 \leq i \leq n$. Then $\theta_1, \theta_2, \ldots, \theta_n$ are true in $\langle Re, \mathscr{F} \cup \{b'\}\rangle$ where $\mathbf{b}$ is interpreted as b' and for each $R \in \mathscr{F}$, $\mathbf{R}$ is interpreted as R. By the compactness theorem (*Theorem 1.1*), let $\langle *Re, *\mathscr{F} \cup \{b\}\rangle$ be a relational system of Σ. Let f be the following function from Re into $*Re$: for all $a \in Re$, $f(a) = c$ where c is the interpretation of $\mathbf{a}$ in $*Re$. Since each sentence of Γ is true in $\langle *Re, *\mathscr{F}\rangle$, it is easy to show that f is an isomorphic embedding of $\langle Re, \mathscr{F}\rangle$ into $\langle *Re, *\mathscr{F}\rangle$. We may therefore assume that $Re \subseteq *Re$ and for all $a \in Re$, a is the interpretation of $\mathbf{a}$ in $\langle *Re, *\mathscr{F}\rangle$. Let b be the interpretation of $\mathbf{b}$ in $*Re$. Since $\mathbf{b} \neq \mathbf{a}$ is true in $\langle *Re, *\mathscr{F} \cup \{b\}\rangle$ for each $a \in Re$, it follows that $b \neq a$ for each $a \in Re$, i.e. $b \in *Re - Re$. Thus $\langle *Re, *\mathscr{F}\rangle$ is a nonstandard model of the reals.

Notation. Let $\langle Re, \mathscr{F}\rangle$ be the full relational system of Re and $\langle *Re, *\mathscr{F}\rangle$ be a nonstandard model of the reals. Then for each n-place relation R in $\mathscr{F}$ there is a predicate symbol $\mathbf{R}$ in the language $L(\mathscr{F})$. In the relational system $\langle Re, \mathscr{F}\rangle$, $\mathbf{R}$ is interpreted as R. If R is a 0-ary relation, then it follows from the definition of nonstandard models of the reals that in $\langle *Re, *\mathscr{F}\rangle$ $\mathbf{R}$ is interpreted as R. If R is a n-ary relation, $n \geq 1$, then, by convention, it will be assumed—unless otherwise explicitly stated—that the interpretation of $\mathbf{R}$ in $\langle *Re, *\mathscr{F}\rangle$ is $*R$.

Suppose that R is a n-place relation on Re where $n \geq 1$ and that $R(a_1, \ldots, a_n)$. Then $\mathbf{R}(\mathbf{a}_1, \ldots, \mathbf{a}_n)$ is true in $\langle Re, \mathscr{F}\rangle$. By elementary equivalence, $\mathbf{R}(\mathbf{a}_1, \ldots, \mathbf{a}_n)$ is also true in $\langle *Re, *\mathscr{F}\rangle$. Since in $\langle *Re, *\mathscr{F}\rangle$ $\mathbf{a}_1, \ldots, \mathbf{a}_n$ are interpreted as $a_1, \ldots, a_n$, we can conclude that $*R(a_1, \ldots, a_n)$. But this means that $*R$ is an extension of R, i.e. that $R \subseteq *R$.

Convention. For convenience and clarity, the extensions of arithmetical operations and relations will often be denoted by the same symbol as those operations and relations of which they are extensions. That is, $*<$ will often be written as $<$, $*=$ as $=$, $*+$ as $+$, etc.

Definition 3.2. Let $\langle *Re, *\mathscr{F}\rangle$ be a nonstandard model of $\langle Re, \mathscr{F}\rangle$. An element α in $*Re$ is said to be *infinitesimal* if and only if for each $r \in Re^+$, $|\alpha| < r$.

Theorem 3.2. There is an infinitesimal β in $*Re^+$.

Proof. Since Re is a proper subset of $*Re$, let α be in $*Re$ and not be in Re. Since $0 \in Re$, $\alpha \neq 0$. Since $\alpha \notin Re$, $-\alpha \notin Re$. Since $\forall x(x \neq \mathbf{0} \rightarrow (\mathbf{0} < x \vee \mathbf{0} < -x))$ is a true statement in $L(\mathscr{F})$ of $\langle Re, \mathscr{F}\rangle$ and therefore of $\langle *Re, *\mathscr{F}\rangle$, either $0 < \alpha$ or $0 < -\alpha$. Without loss of generality, suppose that $0 < \alpha$.

Case 1. $\alpha < r$ for each $r \in Re^+$. Then, by definition, α is an infinitesimal and $\alpha \in *Re^+$.

Case 2. $\alpha > r$ for each $r \in Re^+$. Let $\beta = 1/\alpha$. We will show that β is an infinitesimal and $\beta \in *Re^+$. $\forall x(x > \mathbf{0} \rightarrow \mathbf{1}/x > \mathbf{0})$ is a true statement of $\langle Re, \mathscr{F}\rangle$ and is therefore a true statement of $\langle *Re, *F\rangle$. Since $\beta > 0$ this means that $1/\beta > 0$. Hence $\beta \in *Re^+$. Let r be an arbitrary member of Re^+. Since $1/r < \alpha$, it is easy to show that $\beta = 1/\alpha < r$. But this means that β is an infinitesimal.

Case 3. There are r and s in Re^+ such that $r < \alpha < s$. Let $A_1 = \{t \in Re^+ \mid t < \alpha\}$ and $A_2 = \{t \in Re^+ \mid t > \alpha\}$. Then (A_1, A_2) forms a Dedekind cut of Re^+. Let c be the cut number determined by (A_1, A_2). Since $\alpha \notin Re^+$, $a \neq c$. Therefore, either $\alpha - c > 0$ or $c - \alpha > 0$. Without loss of generality, assume that $c - \alpha > 0$. (The case of $\alpha - c > 0$ follows by a similar argument.) Let r be an arbitrary member of Re^+. Let $\beta = c - \alpha$. We will show that $\beta < r$ thus establishing that β is an infinitesimal. Suppose that $r \leq \beta$. We will show a contradiction. Then $r \leq c - \alpha$, i.e. $r + \alpha \leq c$. Therefore, $\alpha + r/2 < c$. Thus $c \in A_2$. Let $d = c - r/2$. Then $\alpha < d$. Thus $d \in A_2$. Since $d < c$, c cannot be the cut number of (A_1, A_2); a contradiction.

Definition 3.3. Let $\langle *Re, *\mathscr{F}\rangle$ be a nonstandard model of the reals and $\alpha \in *Re$. α is said to be *finite* if and only if $|\alpha| < r$ for some $r \in Re$. α is said to be *infinite* if and only if $|\alpha| > r$ for each $r \in Re$.

Theorem 3.3. If $\alpha \in *Re$ and α is finite then there is $r \in Re$ such that $\alpha - r$ is infinitesimal.

Proof. Let $A_1 = \{t \in Re \mid t \leq \alpha\}$ and $A_2 = \{t \in Re \mid t > \alpha\}$. Then (A_1, A_2) forms a Dedekind cut of Re. Let r be the cut number of (A_1, A_2). Then it is easy to show that $\alpha - r$ is infinitesimal.

Theorem 3.4. If $\alpha \in *Re$, $s, r \in Re$, $\alpha - s$ is infinitesimal, and $\alpha - r$ is infinitesimal, then $r = s$.

Proof left to reader.

Definition 3.4. Let $\langle *Re, *\mathscr{F}\rangle$ be a nonstandard model of the reals and $\alpha \in *Re$ and α be finite. Then, by definition, ${}^\circ\alpha$ is the unique $r \in Re$ such that $\alpha - r$ is infinitesimal.

Theorem 3.5. Let α, α_1 be infinitesimal, β, β_1 be finite, and γ, γ_1 be infinite. Then the following are true:

(1) $\alpha + \alpha_1$ is infinitesimal,
(2) $\alpha\beta$ is infinitesimal,
(3) $\beta + \beta_1$ is finite,
(4) if β and β_1 are not infinitesimal, then $\beta\beta_1$ is finite and not infinitesimal,
(5) $\beta + \gamma$ is infinite,
(6) if β is not infinitesimal then $\beta\gamma$ is infinite, and
(7) $\gamma\gamma_1$ is infinite

Proof left to reader.

4. Imbeddings of Ordered Abelian Groups. *Definition 4.1.* Let $\langle A, \precsim, \circ\rangle$ be an ordered abelian group. Let a, b be in A^+. Then a is said *to be commeasurable with b* if and only if one of the following two conditions hold:

(1) $a \precsim b$ and for some $n \in I^+$ $b \precsim na$, or
(2) $b \precsim a$ and for some $n \in I^+$ $a \precsim nb$.

In the ordered abelian group $\langle *Re, \leq, +\rangle$, every pair of finite noninfinitesimal elements of $*Re^+$ are commeasurable. However, if α is a positive infinitesimal, then α and 1 are not commeasurable.

Theorem 4.1. Let $\langle A, \precsim, \circ\rangle$ be an ordered abelian group. Then the relation defined by x being commeasurable with y is an equivalence relation on A^+.

Proof left to reader.

Definition 4.2. The equivalence classes determined by the commeasurability relation are called *commeasurability classes.*

Definition 4.3. Let $\langle A, \precsim, \circ\rangle$ be an ordered abelian group. Then, by definition, $\hat{A}$ is the set of commeasurability classes of A^+. Also, if U, V are in $\hat{A}$ then, by definition, $U \precsim V$ iff for some $u \in U$ and some $v \in V$, $u \precsim v$.

Theorem 4.2. Let A and $\precsim$ be as in *Definition 4.3.* Then $\precsim$ is a total ordering on $\hat{A}$.

Proof left to reader.

Theorem 4.3. Let $\langle A, \precsim, \circ\rangle$ be an Archimedean ordered group. Then $\hat{A} = \{A^+\}$.

Proof left to reader.

The following theorem shows that the commeasurability classes need not be discretely ordered.

Theorem 4.4. Let A, B be commeasurability classes of $\langle *Re^+, \leq, +\rangle$ such that $A < B$. Then there is a commeasurability class C such that $A < C < B$.

Proof. Let $\alpha \in A$ and $\beta \in B$. Then for some $\gamma \in *Re^+$, $\alpha\gamma = \beta$. Since $A < B$, $\alpha < \beta$. Therefore $\gamma > 1$. Consider $\delta = \alpha\sqrt{\gamma}$. Since $\gamma > 1$, $\alpha < \delta < \beta$. Let n be an arbitrary member of I^+. Since $A < B$, $n^2\alpha < \beta$. Therefore $n^2\alpha < \gamma\alpha = \beta$. Thus $n < \gamma = \sqrt{\gamma}$. Therefore $n\alpha < \sqrt{\gamma}\alpha = \delta$. Since n is an arbitrary member of I^+ we have shown that δ is not in the commeasurability class of α, i.e., $\delta \notin A$. Since $n < \sqrt{\gamma}$ for each $n \in I^+$, $n\delta < \sqrt{\gamma}\delta = \gamma\alpha = \beta$ for each $n \in I^+$. Thus $\delta \notin B$. Let C be the commeasurability class that contains δ. Then $A < C$ and $C < B$.

Let $\langle A, \precsim, \circ\rangle$ be a relational system where $\circ$ is a two place partial operation. Traditionally, $\langle A, \precsim, \circ\rangle$ is said to be an *empirical measurement system* if and only if there is a function G from A into Re^+ that satisfies the following two conditions:

(1) for each x, y in A, $x \prec y$ iff $G(x) < G(y)$, and
(2) for each x, y in A, $G(x \circ y) = G(x) + G(y)$.

Since the function G imbeds A into Re while preserving the intrinsic properties of $\precsim$ and $\circ$, many of the rich algebraic and topological properties of Re can be used for the analysis of the structure of $\langle A, \precsim, \circ\rangle$. However, to guarantee that such a function G exists, it is necessary to assume (perhaps implicitly) some Archimedean axiom.

In the following it will be shown that 'empirical measurement systems' can be adequately axiomatized without use of Archimedean axioms. These measurement systems will be imbedded in structures that are generalizations of the reals. Some of these, $*Re$ for example, will have all the relational and algebraic properties of the reals.

Definition 4.4. Let $\langle A, \precsim, \circ\rangle$ be an ordered abelian group and $S \subseteq A^+$. Then a function f is said to be an *imbedding of S* (with respect to $\precsim, \circ$) into Re^+ (respectively $*Re^+$) if and only if the following three conditions hold:

(1) f is a function from S into Re^+ (respectively $*Re^+$);
(2) for all $x, y \in S$, $x < y$ iff $f(x) < f(y)$; and
(3) for all $x, y \in S, f(x \circ y) = f(x) + f(y)$.

Theorem 4.5. Let $\langle A, \precsim, \circ\rangle$ be an ordered abelian group, $S \subseteq A^+$, and f an imbedding of S into Re^+ or $*Re^+$. Then the following two statements are true:

(1) if $x, y \in S$ and $x \sim y$ then $f(x) = f(y)$, and
(2) if $x \in S$ and $n \in I^+$ then $f(nx) = nf(x)$.

Proof left to reader.

Theorem 4.6. (Hölder's Theorem). Let $\langle A, \precsim, \circ\rangle$ be an Archimedean ordered abelian group. Then there is an imbedding of A^+ into Re.

Proof. See [4], Chapter 2.

Definition 4.5. An ordered abelian group $\langle A, \precsim, \circ\rangle$ is said to be *regularly dense* if and only if for each $x \in A$ and each $n \in I^+$ there is a $y \in A$ such that $x \sim ny$.

Theorem 4.7. (Abraham Robinson and Elias Zakon). Let $\langle A, \precsim, \circ\rangle$ be a regularly dense, ordered abelian group. Let $\mathscr{F} = \{\precsim, \circ\}$. Then there is an Archimedean ordered abelian group that is elementarily equivalent to $\langle A, \precsim, \circ\rangle$ in the language $L(\mathscr{F})$.

Proof. See [13].

Theorem 4.8. Let $\langle A, \precsim, \circ\rangle$ be a regularly dense, ordered abelian group. Then there is a nonstandard model of the reals $\langle *Re, *\mathscr{F}\rangle$, an element u of $*Re^+$, and an imbedding G from A^+ into $*Re^+$ such that for each x in A, $u < G(x)$.

Proof. Let $\mathscr{G} = \{\precsim, \circ\}$. By *Theorem 4.7*, let $\langle B, \precsim_1, \circ_1\rangle$ be an Archimedean ordered group that is elementarily equivalent in the language $L(\mathscr{G})$ to $\langle A, \precsim, \circ\rangle$. Let $D = B \cup Re$. Let $\langle D, \mathscr{H}\rangle$ be the full relational system of D. For notational simplicity, we will assume that $D \cap A = \phi$. Construct the first order language L_1 from $L(\mathscr{H})$ as follows: in each formula of $L(\mathscr{H})$ replace each occurence of $\precsim_1$ by $\boldsymbol{\precsim}$ and each occurence of $\boldsymbol{\circ}_1$ by $\boldsymbol{\circ}$. Naturally, when interpreting the predicate symbols of L_1 in $\langle D, \mathscr{H}\rangle$, $\boldsymbol{\precsim}$ is interpreted as $\precsim_1$ and $\boldsymbol{\circ}$ as $\circ_1$. Let $A_1 = \{\mathbf{a} \mid a \in A\}$ be a new set of individual constant symbols and $\mathbf{c}$ still another new individual constant symbol. (In particular, $\mathbf{c} \notin A_1$.) Let L_2 be the first order language that has as its n-place predicate symbols ($n \geq 1$) the n-place predicate symbols of L_1, and as its individual constant symbols the individual constant symbols of L_1 together with A_1 and $\{\mathbf{c}\}$. Let Γ_0 be the set of sentences of L_1 that are true in $\langle D, \mathscr{H}\rangle$. Let e be an identity element of $\langle A, \precsim, \circ\rangle$. (Recall that the two place operation $\circ$ is interpreted as a three place relation so that $x \circ y = z$ stands for $\circ(x, y, z)$.) Let

$$\Gamma_1 = \{\mathbf{x} \prec \mathbf{y} \mid x, y \in A \text{ and } x \prec y\} \cup \{\mathbf{x} \circ \mathbf{y} = \mathbf{z} \mid x, y, z \in A \text{ and } x \circ y = z\}.$$

Let

$$\Gamma_2 = \{\mathbf{c} \prec \mathbf{x} \mid x \in A^+\} \cup \{\mathbf{e} \prec \mathbf{c}\}.$$

Let

$$\Gamma = \Gamma_0 \cup \Gamma_1 \cup \Gamma_2.$$

We will now show that Γ is finitely satisfiable.

Let Δ be a finite subset of Γ. Let $\theta_1, \ldots, \theta_m$ be the sentences of $\Delta \cap (\Gamma_1 \cup \Gamma_2)$. Let $\mathbf{a}_1, \ldots, \mathbf{a}_n$ be the individual constant symbols of A_1 that occur in the formula $\theta_1\Lambda, \ldots, \Lambda\theta_m$. Let $\Psi(\mathbf{c}, \mathbf{a}_1, \ldots, \mathbf{a}_n)$ be the formula $\theta_1\Lambda, \ldots, \Lambda\theta_m$. Let $a \in A^+$ be such that $a < a_i$ for $i = 1, \ldots, n$. (This can be done since $\langle A, \precsim, \circ\rangle$ is regular.) Then $\Psi(\mathbf{a}, \mathbf{a}_1, \ldots, \mathbf{a}_n)$ is true in $\langle A, \precsim, \circ\rangle$. Therefore $\exists x \exists x_1, \ldots, \exists x_n \Psi(x, x_1, \ldots, x_n)$ is true in $\langle A, \precsim, \circ\rangle$. By elementary equivalence, $\exists x \exists x_1, \ldots, \exists x_n \Psi(x, x_1, \ldots, x_n)$ is also true in $\langle B, \precsim_1, \circ_1\rangle$. Therefore let $b, b_1, \ldots, b_n$ be elements of B such that $\Psi(b, b_1, \ldots, b_n)$ is true in $\langle D, \mathscr{H}\rangle$. Then Δ is simultaneously satisfiable in $\langle D, \mathscr{H}\rangle$ under the following interpretation: each predicate symbol of $\Delta \cap \Gamma_0$ is given its natural interpretation in $\mathscr{H}$; $\precsim$ and $\circ$ are interpreted as $\precsim_1$ and $\circ_1$ respectively; $\mathbf{a}_1, \ldots, \mathbf{a}_n$ are interpreted as $b_1, \ldots, b_n$ respectively; and $\mathbf{c}$ is interpreted as b.

By the compactness theorem (*Theorem 1.1*), Γ is simultaneously satisfiable in some relational system $\mathscr{D}$, $\mathscr{D} = \langle *D, *\precsim, *\circ, A_2, *\mathscr{H}\rangle$, where A_2 is such that $x \in A_2$ iff x is the interpretation of some $\mathbf{y} \in A_1$. Let f be the following function from A^+ onto A_2: $f(x) = y$ iff y is the interpretation in $\mathscr{D}$ of $\mathbf{x}$. Since for each x, y, z in A^+, $x \circ y = z$ iff $\mathbf{x} \circ \mathbf{y} = \mathbf{z}$ is in Γ_1 iff $\mathbf{x} \circ \mathbf{y} = \mathbf{z}$ is true in $\mathscr{D}$ iff $f(x) *\circ f(y) = f(z)$, and $x \prec y$ iff $\mathbf{x} \prec \mathbf{y}$ is in Γ_1 iff $\mathbf{x} \prec \mathbf{y}$ is true in $\mathscr{D}$ iff $f(x) *\prec f(y)$, we can conclude that f is an isomorphic imbedding of A^+ into A_2. Let c be the interpretation of $\mathbf{c}$ in $\mathscr{D}$. Since $\mathscr{D}$ is a relational system for Γ_2, for each $x \in A_2$, $c *\prec x$. Let $*B$ be the interpretation of $\mathbf{B}$ in $\mathscr{D}$. Since $\mathscr{D}$ is a relational system for $\Gamma_1 \cup \Gamma_2$, $A_2 \subseteq *B^+$ and $c \in *B^+$. Since $\langle B, \precsim_1, \circ_1\rangle$ is an Archimedean ordered group, by *Theorem 4.6*, let F be an imbedding of $\langle B, \precsim_1, \circ_1\rangle$ into $\langle Re, \leq, +\rangle$. Then $F \in \mathscr{H}$. Therefore the following sentences are in Γ_0:

(1) $\forall x(\mathbf{B}^+(x) \Leftrightarrow \exists y(\mathbf{Re}^+(y) \wedge \mathbf{F}(x) = y))$,
(2) $\forall x \forall y((\mathbf{B}^+(x) \wedge \mathbf{B}^+(y)) \Rightarrow \mathbf{F}(x \circ y) = \mathbf{F}(x) + \mathbf{F}(y))$,
(3) $\forall x \forall y((B^+(x) \wedge \mathbf{B}^+(y)) \Rightarrow (x \prec y \Leftrightarrow \mathbf{F}(x) \prec \mathbf{F}(y)))$.

Since $\mathscr{D}$ is a relational system for Γ_0, the above three sentences are true in $\mathscr{D}$. Let $*F$ be the interpretation of $\mathbf{F}$ in $\mathscr{D}$. Then,

(1′) for each $x \in *B^+$ there is a $y \in *Re^+$ such that $*F(x) = y$;
(2′) for each x, y in $*B^+$, $*F(x *\circ y) = *F(x) + *F(y)$; and
(3′) for each x, y in $*B^+$, $x *\prec y$ iff $*F(x) < *F(y)$.

Recall that f is an imbedding of $\langle A^+, \precsim, \circ\rangle$ into $\langle *B^+, *\precsim, *\circ\rangle$. For each $x \in A^+$ let$G(x) = *F(f(x))$. Then G is an imbedding of $\langle A^+, \precsim, \circ\rangle$ into $\langle *Re^+, \precsim, \circ\rangle$ Since $c *\prec x$ for each $x \in A_2$, $*F(c) < G(x)$ for each $x \in A^+$.

Definition 4.6. Let $\langle A, \precsim, \circ\rangle$ be an ordered abelian group and $S \subseteq A^+$. Then B is said to be a *set of units for* S if and only if the following three conditions hold:

(1) $B \subseteq S$;
(2) for each $x \in S$ there is a $y \in B$ such that y is commeasurable with x; and
(3) if x and y are in B and $x \neq y$ then x and y are not commeasurable.

Theorem 4.9. Let $\langle A, \precsim, \circ\rangle$ be an ordered abelian group and $S \subseteq A^+$. Then there is a set of units for S.

Proof left to reader.

Definition 4.7. Let $\langle A, \precsim, \circ\rangle$ be an ordered abelian group and $S \subseteq A^+$. A *scale s for S* is a function from S into Re^+ such that the following five conditions hold:

(1) if $x, y \in S$ and x is commeasurable with y then $s(x \circ y) = s(x) + s(y)$;

(2) if $x, y \in S$ and x is commeasurable with y then: (i) if $x \precsim y$ then $s(x) \leq s(y)$, and (ii) if $s(x) < s(y)$ then $x \prec y$;

(3) if $x, y \in S$, $x \prec y$, and x is not commeasurable with y, then $s(x \circ y) = s(y)$;

(4) if $x, y \in S$, x is commeasurable with y, $z \in A^+$, $x \circ z \sim y$, and $s(x) = s(y)$, then z is not commeasurable with x; and

(5) $B = \{x \mid x \in S \text{ and } s(x) = 1\}$ is a set of units for S.

B is called *the set of units for s.*

Theorem 4.10. Let $\langle A, \precsim, \circ\rangle$ be a regularly dense ordered abelian group, $S \subseteq A^+$, and B a set of units for S. Then there is a scale s for S such that B is the set of units for s.

Proof. Let $\langle Re, \mathscr{F}\rangle$ be the full relational system for Re. By *Theorem 4.8* let $\langle *Re, *\mathscr{F}\rangle$, α and G be such that the following four conditions hold:

(1′) $\langle *Re, *\mathscr{F}\rangle$ is an elementary extension of $\langle Re, \mathscr{F}\rangle$;
(2′) $\alpha \in *Re^+$;
(3′) G is an imbedding of A^+ into $*Re^+$; and
(4′) for each $x \in A^+$, $G(x) > \alpha$.

The following sentence of $L(\mathscr{F})$ is true in $\langle Re, \mathscr{F}\rangle$:

$$\forall x \forall y((\mathbf{Re}^+(x) \wedge \mathbf{Re}^+(y) \wedge x \mathbf{<} y) \rightarrow \exists z(\mathbf{I}^+(z) \wedge zx \mathbf{\leq} y \wedge (z \mathbf{+} 1)x \mathbf{>} y)).$$

Therefore, by elementary equivalence, for each x, y in $*Re^+$, if $x < y$ then for some z in $*I^+$, $zx \leq y$ and $(z+1)x > y$. Therefore, for each $x \in S$, let N_x be a member of $*I^+$ such that $N_x\alpha \leq G(x)$ and $(N_x + 1)\alpha > G(x)$. Let a, b be members of S. We will show that the following two statements are equivalent:

(i) a and b are commeasurable,
(ii) N_a/N_b is finite and not infinitesimal.

Without loss of generality, suppose that $a \precsim b$. Suppose (i). Let $n \in I^+$ be such that $na \precsim b$ and $(n + 1)a \succ b$. Then

$$N_b\alpha \le G(b) < G((n + 1)a) = (n + 1)G(a) < (n + 1)(N_a + 1)\alpha.$$

Thus $N_b\alpha < (n + 1)(N_a + 1)\alpha$. Dividing by α we get $N_b < (n + 1)(N_a + 1)$. Thus

$$\frac{1}{n + 1} \le \frac{N_a + 1}{N_b} \le \frac{2N_a}{N_b}.$$

That is,

$$\frac{1}{2(n + 1)} < \frac{N_a}{N_b}.$$

Thus we have shown that N_a/N_b is not infinitesimal. Since $a \precsim b$, it follows that $N_a \le N_b$, i.e. $N_a/N_b \le 1$, i.e. N_a/N_b is finite. We have therefore shown that (i) implies (ii). Now suppose (ii). Since N_a/N_b is not infinitesimal, let $q \in I^+$ be such that $1/q < N_a/N_b$. Then $(N_b + 1) < (q + 1)N_a$. Therefore

$$G(b) < (N_b + 1)\alpha < (q + 1)N_a\alpha \le (q + 1)G(a) = G((q + 1)a).$$

Therefore $b \prec (q + 1)a$. That is, a is commeasurable with b. Therefore (ii) implies (i).

We need the following two lemmas to complete the proof:

Lemma 1: if $x \in A^+$ then N_x is infinite.

Proof: Let $x \in A^+$ and n be an arbitrary member of I^+. Since $\langle A, \precsim, \circ\rangle$ is regularly dense, let $z \in A^+$ be such that $nz \sim x$. Since $z \in A^+$, $\alpha < G(z)$. Therefore $n\alpha < nG(z) = G(nz) = G(x)$. That is, for each $n \in I^+, n\alpha < G(x)$. Therefore for each $n \in I^+, n < N_x + 1$, i.e. N_x is infinite.

Lemma 2: if $x, y \in A^+$ then $N_x + N_y - 1 < N_{x \circ y} < N_x + N_y + 2$.

Proof:

$$N_x\alpha + N_y\alpha \le G(x) + G(y) = G(x \circ y) < (N_{x \circ y} + 1)\alpha$$

and

$$N_{x \circ y}\alpha \le G(x \circ y) = G(x) + G(y) < (N_x + 1)\alpha + (N_y + 1)\alpha = (N_x + N_y + 2)\alpha.$$

For each $x \in S$ let $\beta(x)$ be the unit of B that is commeasurable with x. For each $x \in S$ let

$$s(x) = {}^{\circ}\left(\frac{Nx}{N_{\beta(x)}}\right).$$

We now show that s is a scale for S. Since $\beta(x)$ is commeasurable with x, $N_x/N_{\beta(x)}$ is finite but not infinitesimal. Therefore,

$$^{\circ}\left(\frac{N_x}{N_{\beta(x)}}\right) \in Re^+$$

for each $x \in S$. That is, s is a function from S into Re^+.

(1) Suppose $x, y \in S$ and x is commeasurable with y. Then $x \circ y$ is commeasurable with x. Therefore $\beta(x) = \beta(y) = \beta(x \circ y)$. Let $N = N_{\beta(x)}$. Then by *Lemma 1*, N is infinite. By *Lemma 2*,

$$\frac{N_x + N_y - 1}{N} < \frac{N_{x \circ y}}{N} < \frac{N_x + N_y + 2}{N}.$$

Since N is infinite, $1/N$ and $2/N$ are infinitesimal. Therefore by simple algebra

$$s(x) + s(y) = {}^{\circ}\left(\frac{N_x + N_y}{N}\right) = {}^{\circ}\left(\frac{N_x + N_y - 1}{N}\right) \leq {}^{\circ}\left(\frac{N_{x \circ y}}{N}\right) = s(x \circ y)$$

$$\leq {}^{\circ}\left(\frac{N_x + N_y + 2}{N}\right) = {}^{\circ}\left(\frac{N_x + N_y}{N}\right) = s(x) + s(y).$$

That is, $s(x) + s(y) \leq s(x \circ y) \leq s(x) + s(y)$, i.e. $s(x \circ y) = s(x) + s(y)$.

(2) Let $x, y \in S$ be such that x is commeasurable with y. Then $\beta(x) = \beta(y)$. Let $N = N_{\beta(x)}$. Then, (i) if $x \precsim y$ then $N_x \leq N_y$ and ${}^{\circ}(N_x/N) \leq {}^{\circ}(N_y/N)$, i.e. $s(x) \leq s(y)$, and (ii) if $s(x) < s(y)$ then ${}^{\circ}(N_x/N) < {}^{\circ}(N_y/N)$, i.e. $N_x < N_y$, i.e. $x \prec y$.

(3) Let $x, y \in S$ be such that $x \precsim y$ and x is not commeasurable with y. Since $y \prec x + y \prec y + y$, y is commeasurable with $x + y$. Therefore $\beta(y) = \beta(x \circ y)$. Let $z = \beta(y)$. Since x is not commeasurable with y and y is commeasurable with z, x is not commeasurable with z. Since $x \prec y$, it then follows that $x \prec z$. Therefore, for each $n \in I^+$, $nx \prec z$. Thus for each $n \in I^+$, $G(nx) = nG(x) < G(z)$. That is, $nN_x\alpha < (N_z + 1)\alpha$ for each $n \in I^+$. That is, for each $n \in I^+$, $nN_x - 1 < N_z$. Thus

$$\frac{N_x}{N_z} < \frac{1}{n} + \frac{1}{nN_z} < \frac{1}{n - 1}$$

for each $n \in I^+, n \geq 2$. By *Definition 3.2*, this means that N_x/N_z is infinitesimal. Since by (1), $s(x \circ y) = s(x) + s(y)$, we can conclude that

$$s(x \circ y) = s(x) + s(y) = {}^{\circ}\left(\frac{N_x + N_y}{N_z}\right) = {}^{\circ}\left(\frac{N_y}{N_z}\right) = s(y).$$

(4) Let $x, y \in S$ and $z \in A^+$ be such that x is commeasurable with y, $x \circ z \sim y$, and $s(x) = s(y)$. Then we will show by contradiction that z is not commeasurable with x. Assume that z is commeasurable with x. Let $n \in I^+$ be such that $x \precsim nz$. Then $N_x\alpha < n(N_z + 1)\alpha$. That is, $N_x < n(N_z + 1) = nN_z + n$. Thus,

$$\frac{N_x - n}{n} = \frac{1}{n} N_x - 1 < N_z.$$

Since x is commeasurable with y, $\beta(x) = \beta(y)$. Let $N = N_{\beta(x)}$. Then, by *Lemma 2*,

$$N_x + \left(\frac{1}{n} N_x - 1\right) - 1 < N_x + N_z - 1 < N_{x \circ z} = N_y.$$

Since, by *Lemma 1*, $1/N$ is infinitesimal and

$${}^{\circ}\left(\frac{N_y}{N}\right) = s(y) = s(x) = {}^{\circ}\left(\frac{N_x}{N}\right),$$

we can conclude that

$$\left(1+\frac{1}{n}\right){}^{\circ}\left(\frac{N_x}{N}\right) = {}^{\circ}\left(\frac{N_x + \left(\frac{1}{n}N_x - 1\right) - 1}{N}\right) \le {}^{\circ}\left(\frac{N_y}{N}\right) = {}^{\circ}\left(\frac{N_x}{N}\right).$$

That is,

$$\left(1+\frac{1}{n}\right){}^{\circ}\left(\frac{N_x}{N}\right) \le {}^{\circ}\left(\frac{N_x}{N}\right)$$

which is impossible since

$${}^{\circ}\left(\frac{N_x}{N}\right) \in Re^{+}.$$

Theorem 4.11. Let $\langle A, \precsim, \circ\rangle$ be an ordered abelian group, $S \subseteq A^{+}$, and s, t be scales for S. Let T be a subset of S that satisfies the following two conditions:

(1) for each $x, y \in T$, x is commeasurable with y; and
(2) for each $x \in T$, $nx \in T$ for sufficiently large positive integers n.

Then there is a positive real number r such that for all $x \in S$, $s(x) = rt(x)$.

Proof. Let $x \in T$ and $r = s(x)/t(x)$. Let $y \in T$. Then for each sufficiently large $n \in I^{+}$, let $N_n \in I^{+}$ be such that $N_n y \precsim nx \prec (N_n + 1)y$. Then N_n approaches infinity as n approaches infinity. Also $s(N_n y) \le s(nx) \le s((N_n + 1)y)$. From which we conclude, $N_n s(y) \le ns(x) \le (N_n + 1)s(y)$. Therefore

$$\frac{N_n}{n} \le \frac{s(x)}{s(y)} \le \frac{N_n + 1}{n}.$$

Similarly,

$$\frac{N_n}{n} \le \frac{t(x)}{t(y)} \le \frac{N_n + 1}{n}.$$

Letting n approach infinity, we have

$$\lim \frac{N_n}{n} = \frac{s(x)}{s(y)} = \frac{t(x)}{t(y)}.$$

Since $s(x) = rt(x)$, we get

$$\frac{rt(x)}{s(y)} = \frac{t(x)}{t(y)},$$

i.e. $rt(y) = s(y)$.

5. Extensive Structures. *Definition 5.1.* Let $\langle A, \precsim, \circ\rangle$ be a relational system such that $\precsim$ is a two place relation on A, $B \subseteq A \times A$, and $\circ$ is a function from B into A. (I.e. $\circ$ is a partial operation on A.) Then $\langle A, \precsim, \circ\rangle$ is said to be an *extensive structure* if and only if the following six conditions are satisfied for all $a, b, c \in A$:

(1) $\precsim$ is a weak order on A;

(2) if $(a, b) \in B$ and $(a \circ b, c) \in B$, then $(b, c) \in B$, $(a, b \circ c) \in B$, and $(a \circ b) \circ c \succsim a \circ (b \circ c)$;

(3) if $(a, c) \in B$ and $a \succsim b$, then $(c, b) \in B$ and $a \circ c \succsim c \circ b$;

(4) if $a \succ b$, then there exists $d \in A$ such that $(b, d) \in B$ and $a \succsim b \circ d$;

(5) if $(a, b) \in B$, then $a \circ b \succ a$; and

(6) there exist $x, y \in A$ such that $a \precsim x \circ y$.

Definition 5.2. The relational system $\langle A, \precsim, \circ \rangle$ is said to be an *Archimedean extensive structure* if and only if the following two conditions hold:

(1) $\langle A, \precsim, \circ \rangle$ is an extensive structure; and

(2) if $a_1, a_2, \ldots, a_n, \ldots,$ is an (infinite) sequence of members of A such that for $n = 2, 3, \ldots, a_n = a_{n-1} \circ a_1$, then for all $b \in A$ it is not the case that for each $n \in I^+$, $a_n \prec b$.

Theorem 5.1. Let $\langle A, \precsim, \circ \rangle$ be an Archimedean extensive structure and B be the domain of $\circ$. Then there is a function f from A into Re^+ such that the following two conditions hold for all $x, y \in A$:

(1) $x \prec y$ iff $f(x) < f(y)$; and

(2) if $(x, y) \in B$ then $f(x \circ y) = f(x) + f(y)$.

Furthermore, if g is another function from A into Re^+ that satisfies (1) and (2), then there is $r \in Re^+$ such that for all $z \in A$, $g(z) = rf(z)$.

Proof. See [4], Chapter 3.

Theorem 5.2. Let $\langle A, \precsim, \circ \rangle$ be an ordered abelian group, $S \subseteq A^+$, and s, t scales for S. Let $a \in S$ and $T = \{y \in S \mid y \text{ is commeasurable with } a\}$. Suppose that T satisfies the following three conditions:

(1) if $x, y, z \in T$, $x \circ y \in T$, and $z \precsim y$, then $x \circ z \in T$;

(2) if $x, y \in T$ and $x \prec y$, then there is a $z \in T$ such that $x \circ z \in T$ and $x \prec x \circ z \precsim y$; and

(3) if $x \in T$ then there are $y, z \in T$ such that $x \precsim y \circ z$.

Then there is a positive real number r such that for all $x \in T$, $s(x) = rt(x)$.

Proof. Let $\precsim_1$ and $\circ_1$ be the restrictions of $\precsim$ and $\circ$ to T. Then it is easy to verify that $\langle T, \precsim_1, \circ_1 \rangle$ is an extensive structure. Since all members of T are commeasurable, $\langle T, \precsim_1, \circ_1 \rangle$ is Archimedean. Therefore, by *Theorem 5.1*, there is a positive real number r such that for all $x \in T$, $s(x) = rt(x)$.

Definition 5.3. The relational system $\langle A, \precsim, \circ \rangle$ is said to be a *closed extensive structure* if and only if the following five conditions hold for all $a, b, c \in A$:

(1) $\precsim$ is a weaker order on A;

(2) $\circ$ is a binary operation on A;

(3) $(a \circ b) \circ c \succsim a \circ (b \circ c)$;

(4) if $a \succsim b$ then $a \circ c \succsim c \circ b$; and

(5) $a \circ b \succ a$.

Theorem 5.3. Let $\langle A, \precsim, \circ\rangle$ be a closed extensive structure. Then the following are true for all $a, b, c \in A$:

(1) if $a \succsim c$ and $b \succsim d$ then $a \circ b \succsim c \circ d$;
(2) $a \circ b \sim b \circ a$;
(3) $a \circ (b \circ c) \sim (a \circ b) \circ c$.

Proof left to reader.

Theorem 5.4. Let $\langle A, \precsim, \circ\rangle$ be a closed extensive structure. Let

$D = \{(x, y) \mid x, y \in A\}$,
$P = \{(x, y) \mid x, y \in A \text{ and } x \succsim y\}$,
$\oplus$ be the 2-place operation on D such that $(a, b) \oplus (c, d) = (a \circ c, b \circ d)$,
$(a, b)^{-1} = (b, a)$, and
${}_1\succsim$ be such that $(a, b)\ {}_1\succsim (c, d)$ iff $(a, b) \oplus (c, d) \in P$.

Then $\langle D, \precsim_1, \oplus\rangle$ is an ordered abelian group. Furthermore, $\langle A, \precsim, \circ\rangle$ is isomorphic to a subset of D^+. (That is, there is a one-to-one function f, $f(x) = (2x, x)$, from A into D^+ such that (1) $x \precsim y$ iff $f(x) \precsim_1 f(y)$ and (2) $f(x \circ y) = f(x) \oplus f(y)$.)

Proof left to reader.

Theorem 5.5. Every ordered abelian group is a subgroup of a regularly dense ordered abelian group.

Proof. Let $\langle A, \precsim, \circ\rangle$ be an ordered abelian group. Let e be an identity element of A. For each $x \in A$, let x^{-1} be an element of A such that $x \circ x^{-1} \sim e$. For each $n \in I^+$ and each $x \in A$, define $(-n)x$ to be nx^{-1}. Let $B = \{(x, n) \mid x \in A \text{ and } n \in I - \{0\}\}$. Define the two place operation $\oplus$ on B as follows:

$$(x, n) \oplus (y, m) = ((mx) \circ (ny), nm).$$

Define the two place relation $\precsim_1$ on B as follows: $(x, n) \precsim_1 (y, m)$ if and only if one of the following two conditions hold:

(1) nm is positive and $e \precsim ny \circ (-m)x$, or
(2) nm is negative and $e \succsim ny \circ (-m)x$.

It is easy to verify that $\langle B, \precsim_1, \oplus\rangle$ is an ordered abelian group. It is easy to show that for each $n \in I^+$ and each $(x, m) \in B$ that $n(x, nm) = (x, m)$. Thus $\langle B, \precsim, \oplus\rangle$ is regularly dense. One can also verify that the function f, defined by $f(x) = (x, 1)$ iff $x \in A$, is an isomorphic imbedding of $\langle A, \precsim, \circ\rangle$ into $\langle B, \precsim_1, \oplus\rangle$. We may therefore consider $\langle A, \precsim, \circ\rangle$ as a subgroup of $\langle B, \precsim_1, \oplus\rangle$.

Definition 5.4. Let $\langle A, \precsim_1, \circ_1\rangle$ be a relational system where $\precsim_1$ is a binary relation on A and $\circ_1$ is a binary partial operation on A (i.e. $\circ_1$ is a function from $D \times D$ into A for some $D \subseteq A$). Then $\langle B, \precsim, \circ\rangle$ is said to be a *regularly dense ordered abelian group extension of* $\langle A, \precsim_1, \circ_1\rangle$ if and only if $A \subseteq B$, $\precsim_1 \subseteq \precsim$, $\circ_1 \subseteq \circ$, and $\langle B, \precsim, \circ\rangle$ is a regularly dense ordered abelian group.

Theorem 5.6. Let $\langle A, \precsim_1, \circ_1\rangle$ be a closed extensive structure. Then there is a regularly dense ordered abelian group extension $\langle B, \precsim, \circ\rangle$ of $\langle A, \precsim_1, \circ_1\rangle$ such that $A \subseteq B^+$.

Proof. Theorem 5.4 and *Theorem 5.5.*

Theorem 5.7. Let $\langle A, \precsim_1, \circ_1\rangle$ be a closed extensive structure. Then there is a nonstandard model of the reals $\langle *Re, *\mathscr{F}\rangle$ and a function f from A into $*Re^+$ such that

(1) $x \prec_1 y$ iff $f(x) < f(y)$; and
(2) $f(x \circ_1 y) = f(x) + f(y)$.

Proof. By *Theorem 5.6*, let $\langle B, \precsim, \circ\rangle$ be a regularly dense ordered abelian group extension of $\langle A, \precsim_1, \circ_1\rangle$. Since $A \subseteq B^+$, by *Theorem 4.8*, let $\langle *Re, *\mathscr{F}\rangle$ and f be such that f is an imbedding of A into $*Re^+$. Then f has the required properties (1) and (2).

Definition 5.5. Let $\langle A, \precsim, \circ\rangle$ be a closed extensive structure and $x, y \in A$. Let $1x = x$ and for each $n \in I^+$, $(n + 1) = (nx) \circ x$. x is said to be *commeasurable with* y if and only if (1) $x \precsim y$ and for some $n \in I^+$, $nx \succsim y$, or (2) $y \precsim x$ and for some $n \in I^+$, $ny \succsim x$. As before, 'is commeasurable with' can be shown to be an equivalence relation. A function s from A into Re^+ is said to be a *closed extensive scale for* $\langle A, \precsim, \circ\rangle$ if and only if the following four conditions hold for all x, y, z in A:
(1) if x is commeasurable with y then $s(x \circ y) = s(x) + s(y)$;
(2) if x is commeasurable with y then:

(i) if $x \precsim y$ then $s(x) \leq s(y)$, and
(ii) if $s(x) < s(y)$ then $x \prec y$;

(3) if $x \prec y$ and x is not commeasurable with y, then $s(x \circ y) = s(y)$; and
(4) if x and y are commeasurable, $s(x) = s(y)$, and $x \circ z \precsim y$, then z is not commeasurable with x.

Theorem 5.8. Let $\langle A, \precsim, \circ\rangle$ be a closed extensive structure. Then there is a closed extensive scale for $\langle A, \precsim, \circ\rangle$.

Proof. Theorem 5.6 and *Theorem 4.10.*

Theorem 5.9. Let $\langle A, \precsim, \circ\rangle$ be a closed extensive structure, s and t be closed extensive scales for $\langle A, \precsim, \circ\rangle$, and $a \in A$. Let $T = \{x \in A \mid x$ is commeasurable with $a\}$. Then there is a positive real number r such that for each $x \in T$, $s(x) = rt(x)$.

Proof. Theorem 5.6 and *Theorem 4.11.*

Definition 5.6. Let $\langle A, \precsim, \circ\rangle$ be an extensive structure, B be the domain of $\circ$ and $x \in A$. Then, by definition $1x = x$. Furthermore, if nx has been defined for $n \in I^+$ and $(nx, x) \in B$ then, by definition, $(n + 1)x = (nx) \circ x$. If $(nx, x) \notin B$ then $(n + 1)x$ is not defined.

Definition 5.7. Let $\langle A, \precsim, \circ\rangle$ be an extensive structure. An element $x \in A$ is said to be small if and only if for some $y \in A$, $mx \prec y$ for each $m \in I^+$.

Theorem 5.10. Let $\langle A, \precsim, \circ \rangle$ be an extensive structure. Suppose $x \precsim y$ and y is small. Then for each $n \in I^+$, nx is small.

Proof. Let B be the domain of $\circ$. We will show by induction that nx is defined for each $n \in I^+$ and that $nx \precsim ny$. By definition, $1x$ is defined and $1x \precsim 1y$. Let $p \in I^+$ and suppose that px has been defined and that $px \precsim py$. Since $(py, y) \in B$ and $py \succsim px$, by (3) of *Definition 5.1*, $(y, px) \in B$ and $(p + 1)y = (py) \circ y \succsim y \circ (px)$. Since $(y, px) \in B$ and $y \succsim x$, by (3) of *Definition 5.1*, $(px, x) \in B$ and $y \circ (px) \succsim (px) \circ x = (p + 1)x$. Thus we have shown that $(p + 1)x$ is defined and $(p + 1)x \precsim (p + 1)y$. Thus by induction, for each $n \in I^+$, $nx \precsim ny$. Since for some $z \in A$, $ny \prec z$ for each $n \in I^+$, $nx \prec z$ for each $n \in I^+$. That is, x is small. Let $m \in I^+$ and $nx \prec z$ for each $n \in I^+$. Then $n(mx) = (nm)x \prec z$ for each $n \in I^+$. That is, mx is small for each $m \in I^+$.

Theorem 5.11. Let $\langle A, \precsim, \circ \rangle$ be an extensive structure and $D = \{x \mid x \in A$ and x is small$\}$. Suppose that $D \neq \phi$. Let $\precsim_1$ and $\circ_1$ be the restrictions of $\precsim$ and $\circ$ to D. Then $\langle D, \precsim_1, \circ_1 \rangle$ is a closed extensive structure.

Proof. We need only show that $\circ_1$ is an operation on D since all the other conditions for a closed extensive structure follow immediately from the fact that $\langle A, \precsim, \circ \rangle$ is an extensive structure and from the definitions of $\precsim_1$ and $\circ_1$. Let B be the domain of $\circ$ and let x, y be elements of D. To show that $\circ_1$ is an operation on D we need only show that $(x, y) \in B$ and $x \circ y \in D$.

Case 1. Suppose that $x \precsim y$. Since $(y, y) \in B$ and $x \precsim y$, we have by condition (3) of *Definition 5.1* that $(x, y) \in B$ and $x \circ y \precsim y \circ y$. Since $y \circ y \in D$, by *Theorem 5.10*, $x \circ y \in D$.

Case 2. Suppose that $x \succ y$. Then by Case 1 $(y, x) \in B$. Since $y \precsim y$, by Condition (3) of *Definition 5.1*, $(x, y) \in B$. Since $x \circ y \precsim x \circ x$ and $x \circ x \in D$, by *Theorem 5.10*, $x \circ y \in D$.

Definition 5.8. Let $\langle A, \precsim, \circ \rangle$ be an extensive structure and $x \in A$. Then x is said to be *large* if and only if x is not small.

Theorem 5.12. Let $\langle A, \precsim, \circ \rangle$ be an extensive structure, $x, y \in A$, and x be small and y be large. Then $x \prec y$.

Proof. Since $\precsim$ is a weak ordering, either $x \prec y$ or $y \precsim x$. Since y is large, by *Theorem 5.10*, it cannot be the case that $y \precsim x$. Therefore $x \prec y$.

In order to assure that an extensive structure can be represented, it is necessary to add conditions on the large elements to assure that they can be "measured." This is done in the following definition.

Definition 5.9. An extensive structure $\langle A, \precsim, \circ \rangle$ is said to be *good* if and only if for all large elements a, b of A the following two conditions hold:

(1) if $a \prec b$ and for all small α, $a \circ \alpha \prec b$, then for some large c in A, $a \circ c \precsim b$; and

(2) there are large c and d in A such that $a \prec c \circ d$.

Definition 5.10. Let $\langle A, \precsim, \circ \rangle$ be a good extensive structure and D be the set of large elements of A. Then the function f from D into Re^+ is said to be an *extensive* *imbedding of D into* Re^+ if and only if the following three conditions hold for all $x, y \in A$:

(1) if (x, y) is the domain of $\circ$ then $f(x \circ y) = f(x) + f(y)$;

(2) $f(x) < f(y)$ iff for some $z \in D$, $x \circ z \precsim y$; and

(3) if $x \precsim y$ then: $f(x) = f(y)$ iff either $x \sim y$ or for some small α in A, $x \circ \alpha \succsim y$.

Theorem 5.13. Let $\langle A, \precsim, \circ \rangle$ be a good extensive structure and D be the set of large elements of A. Then there is an extensive imbedding of D into Re^+.

Outline of proof. By Zorn's lemma, let D' be a maximal subset of D such that if $x, y \in D'$ and $x \prec y$ then for some large $z \in D$, $x \circ z \precsim y$. Define the relation E on D as follows: xEy if and only if $x \in D'$, $y \in D$ and [($x \prec y$ and for some small $\alpha \in A$, $x \circ \alpha \succsim y$) or ($x \sim y$) or ($y \prec x$ and for some small $\alpha \in A$, $y \circ \alpha \succsim x$)]. Then it is easy to show that for all $y \in D$ there is an $x \in D'$ such that xEy. Define $\circ_1$ and $\precsim_1$ on D' as follows: $x \circ_1 y = z$ iff there are x_1, y_1, z_1 in D such that xEx_1, yEy_1, zEz_1, and $x_1 \circ y_1 = z_1$; and $x \precsim_1 y$ iff for some $x_1, y_1 \in D$, xEx_1, yEy_1, and $x_1 \precsim y_1$. Then one can show that $\langle D', \precsim_1, \circ_1 \rangle$ is an Archimedean extensive structure. By *Theorem 5.1*, let g be an imbedding of $\langle D', \precsim_1, \circ_1 \rangle$ into Re^+. Define the function f from D into Re^+ as follows: if $y \in D$, let $f(y) = g(x)$ where x is such that xEy. Then it is easy to show that f is an extensive imbedding of D into Re^+.

We will now extend the definition of 'commeasurability' and 'scale' to extensive structures.

Definition 5.11. Let $\langle A, \precsim, \circ \rangle$ be an extensive structure and $a, b \in A$. Then a is said to be *extensively commeasurable with b* if and only if (i) a and b are large elements of A, or (ii) a and b are small elements of A and a and b are commeasurable (as defined for closed extensive structures—*Definition 5.5*). A function s from A into Re^+ is said to be an *extensive scale* for $\langle A, \precsim, \circ \rangle$ if and only if the following four conditions hold for all x, y, z in A:

(1) if x is extensively commeasurable with y and $(x, y) \in$ domain $\circ$, then $s(x \circ y) = s(x) + s(y)$;

(2) if x is extensively commeasurable with y then: (i) if $x \precsim y$ then $s(x) \leq s(y)$, and (ii) if $s(x) < s(y)$ then $x \prec y$;

(3) if $x \prec y$ and x is not commeasurable with y, then $s(x \circ y) = s(y)$; and

(4) if x and y are commeasurable, $s(x) = s(y)$, and $x \circ z \precsim y$, then z is not commeasurable with x.

Theorem 5.14. Let $\langle A, \precsim, \circ \rangle$ be an extensive structure, s an extensive scale for $\langle A, \precsim, \circ \rangle$, $a \in A$, and $T = \{x \in A \mid x \text{ is extensively commeasurable with } a\}$. Let $r \in Re^+$ and t be a function from A into Re^+ such that for all $x \in A - T$, $t(x) = s(x)$, and for all $x \in T$, $t(x) = rs(x)$. Then t is an extensive scale for $\langle A, \precsim, \circ \rangle$.

Proof. Immediate from *Definition 5.11*.

Theorem 5.15. Let $\langle A, \precsim, \circ \rangle$ be a good extensive structure. Then there is an extensive scale s for $\langle A, \precsim, \circ \rangle$. Furthermore, if t is another extensive scale for $\langle A, \precsim, \circ \rangle$, $x \in A$, and $T = \{y \in A \mid y \text{ is extensively commeasurable with } x\}$, then there is a $r \in Re^+$ such that for all $z \in T$, $s(z) = rt(z)$.

Proof. Let S be the set of small elements of A and D be the set of large elements of A. if $S = \phi$, then $\langle A, \precsim, \circ\rangle$ is an Archimedean extensive structure and the theorem follows from *Theorem 5.1*. If $D = \phi$, then the theorem follows from *Theorems 5.8* and *5.9*. Therefore, assume that $S \neq \phi$ and $D \neq \phi$. By *Theorem 5.8* let s_1 be a closed extensive scale for S and by *Theorem 5.13* s_2 be an extensive imbedding of D into Re^+. Let $s = s_1 \cup s_2$. Then it is easy to verify that s is an extensive scale for $\langle A, \precsim, \circ\rangle$. It immediately follows from the definition of s and from *Theorems 5.9, 5.1*, and the proof of *Theorem 5.13* that if $x \in A$ and $T = \{y \in A \mid y$ is extensively commeasurable with $x\}$ and t is an extensive scale for $\langle A, \precsim, \circ\rangle$ then for some $r \in Re^+$, $s(z) = rt(z)$ for all $z \in T$.

The relationships between extensive structures and imbeddings into nonstandard models of the reals will be described in the next three theorems. The proofs of these theorems will be omitted.

Theorems 5.16. Let $\langle A, \precsim, \circ\rangle$ be a good extensive structure. Then there is a nonstandard model of the reals $\langle *Re, *\mathscr{F}\rangle$ and a function f such that f is an imbedding of A into $*Re^+$.

The real number system is a *universal* measuring system in the sense that each Archimedean extensive structure can be imbedded into Re^+. So far we have only shown that for each extensive structure there is a nonstandard model of the reals into which it can be imbedded. We have not shown, for example, that two extensive structures can be imbedded in the same nonstandard model of the reals. By an intelligent application of the compactness theorem (*Theorem 1.1*), the following theorem can be proved:

Theorem 5.17. Let $\mathscr{C}$ be a nonempty class of extensive structures. Then there is a nonstandard model of the reals $\langle *Re, *\mathscr{F}\rangle$ such that each member of $\mathscr{C}$ can be imbedded in $*Re^+$.

An even stronger version of *Theorem 3.4* can be proved by using *saturated models*. (See [1], Chapter 11.)

Theorem 5.18. Let $\aleph$ be the cardinality of Re (alternatively, $\aleph$ be a regular uncountable cardinal) and $\mathscr{C}$ a nonempty class of extensive structures such that each member of $\mathscr{C}$ has cardinality $\leq \aleph$. Assume the continuum hypothesis (alternatively, the generalized continuum hypothesis). Then there is a nonstandard model of the reals, $\langle *Re, *\mathscr{F}\rangle$, such that $*Re$ has cardinality $\aleph$ and each member of $\mathscr{C}$ is imbeddable in $*Re^+$.

6. Historical Note. Axiomatic approaches to Archimedean extensive attributes were made by Helmoltz [2] and Hölder [3]. The axioms for Archimedean closed extensive structures are like Robert's and Luce's in [10]. The axioms for Archimedean extensive structures which are due to Krantz, *et. al.* [4], are a modification of axioms given in Luce and Marley [5]. Abraham Robinson has applied the compactness theorem to a wide variety of algebraic problems in Robinson [11], and has used the nonstandard reals for the solution of many problems in Robinson [12]. Representation and uniqueness theorems for non-Archimedean additive conjoint structures and non-Archimedean qualitative probability structures are given in Narens [8], and non-Archimedean expected utility structures in Narens [9].

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